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Periodically kicked Duffing oscillator and nonattracting chaotic sets

Marek Franaszek

Institute of Physics, Cracow Pedagogical University, Podchorążych 2, Cracow, Poland

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The Duffing oscillator perturbed periodically by a sequence of short kicks of a constant strength is investigated. It appears that long time behavior of a such system may be connected with a nonattracting chaotic set which exists in a free unperturbed system. This set corresponds to the unstable periodic orbits (placed on the basins boundaries) and their unstable manifolds.

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High dimensional dissipative systems may possess two or more different attractors coexisting for a given set of control parameters. The attractors may be either stable periodic cycles or chaotic sets. A suitable choice of initial conditions gives us a possibility to observe different kinds of asymptotic behavior for $t \to +\infty$. Each attractor is embedded in its own basin of attraction and each two neighboring basins are separated by a basin boundary. Very often a boundary is a complicated fractal set [1, 2]. The trajectory, which starts exactly from a point belonging to this set, does not approach any attractor but it stays on a boundary forever. However, the probability of finding such a peculiar trajectory in any experiment is equal to zero. It is a direct consequence of the fact that a basin boundary is a nonattracting set [3] (saddle periodic points and their stable manifolds). Any experimental trajectory which starts even very close to this nonattracting set leaves its close neighborhood after a finite time and tends to the corresponding attractor. Thus, the tracing of a single long trajectory is not sufficient in the investigation of a nonattracting set. Instead, we must consider an ensemble of relatively short trajectories and from each trajectory we must cut off a piece properly so that it approximates well a part of the nonattracting set [4]. Then, we can glue together these properly selected pieces of trajectories and we can get a set which is very close to the searched true nonattracting set. This general procedure was successfully applied in a few numerical [5, 6] and experimental [7, 8] investigations.

In spite of the simplicity of the described procedure, much less attention was paid in the past to studying the properties of nonattracting chaotic sets than to the properties of chaotic attractors. One may argue that the attractor is a "natural" asymptotic state of a dissipative dynamical system, while the nonattracting set may be seen only in a specially designed experiment. Thus, the attractor and its properties are much more important. It is not alwayes true. The aim of this paper is to show that in certain circumstances the nonattracting set may be responsible for the long time behavior of the system and the role of the attractor is then reduced significantly.

Such a situation may arise when stochastic noise is added to the purely deterministic dynamics [9–11]. Stochastic perturbation disables a trajectory to reach the final attractor. Noise of sufficiently large amplitude σ may even switch a trajectory between two or more basins of attraction. In this case, a trajectory spends much more time in the neighborhood of the deterministic (for $\sigma = 0$) basin boundaries than in the vicinity of purely deterministic attractors. If the coexisting attractors are simple periodic cycles then a noisy trajectory drastically differs from the noiseless one [11] (for example, the greatest Lyapunov exponent of a noisy system may become positive). Thus, the nonattracting chaotic set which exists for a deterministic system becomes responsible for the asymptotic behavior of a noisy system.

In the current paper we investigate the deterministic dynamical system which is perturbed by a periodic sequence of kicks. The studied system is the Duffing oscillator

$$\ddot{x}(t) + \gamma\dot{x}(t) + \frac{d}{dx}V(x) = \lambda \sin(2\pi t/T) , \qquad (1)$$

where the asymmetric double well potential $V(x)$ is given by

$$V(x) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 - \frac{1}{2}cx^2 . \qquad (2)$$

The parameter b is periodically switching between two constant values: b_0 and $b_0 - \Delta$; see Fig. 1. The period of the two state oscillations is denoted by τ while Δ and $\tau_k = \epsilon T$ stand for the amplitude and the duration of a single kick, respectively. One may expect that for the strength of kicks $\Delta \to 0$ the dynamics of the perturbed system will not differ significantly from the free unperturbed dynamics. On the other hand, for large Δ and short period τ the perturbed system may be quite far from the free system and both dynamics may have nothing in common. The most interesting is the case of intermediate values of the amplitude Δ, the relatively long period τ, and the short duration of kick τ_k. In this range of parameters the free system is perturbed rarely by short lasting pulses. The time interval between two successive kicks is sufficintly long to enable a relaxation of the system to its free dynamics.

The unperturbed Duffing oscillator is studied for the following parameters: $\lambda = 0.7262$, $T = 1.7943$, and $\gamma = 1$ and the potential parameters $a = 10$, $b_0 = 7.5$, and $c = 100$. For these parameters we can observe four

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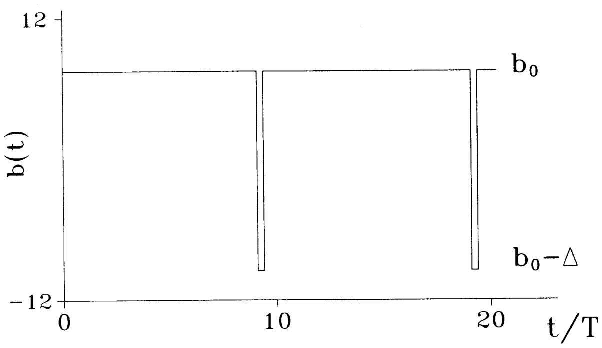

FIG. 1. The two state oscillations of the parameter b [see Eq. (2)], which introduce a perturbation to the Duffing oscillator.

different coexisting attractors. We look at the image of uniformly distributed starting points after time $t = nT$, where $n = 1, 2, ..., N$. More precisely, a surface of section $(x(t), y(t), \phi_0 = 0)$ is plotted, where $y(t) = \dot{x}(t)$ and ϕ_0 is the fixed phase of the sinusoidal drive of period T. A few examples of such images are shown in Fig. 2. The most striking feature of these plots is a very fast contraction of the space of the initial conditions to a certain set. Most of the trajectories visit first this set and spend a finite time in its neighborhood before they reach a particular attractor. More detailed analysis suggests that this nonattracting set is very close to the unstable manifolds of a few unstable periodic points which are placed on the basins boundaries.

The asymptotic behavior of the unperturbed system is connected to four different coexisting attractors; see

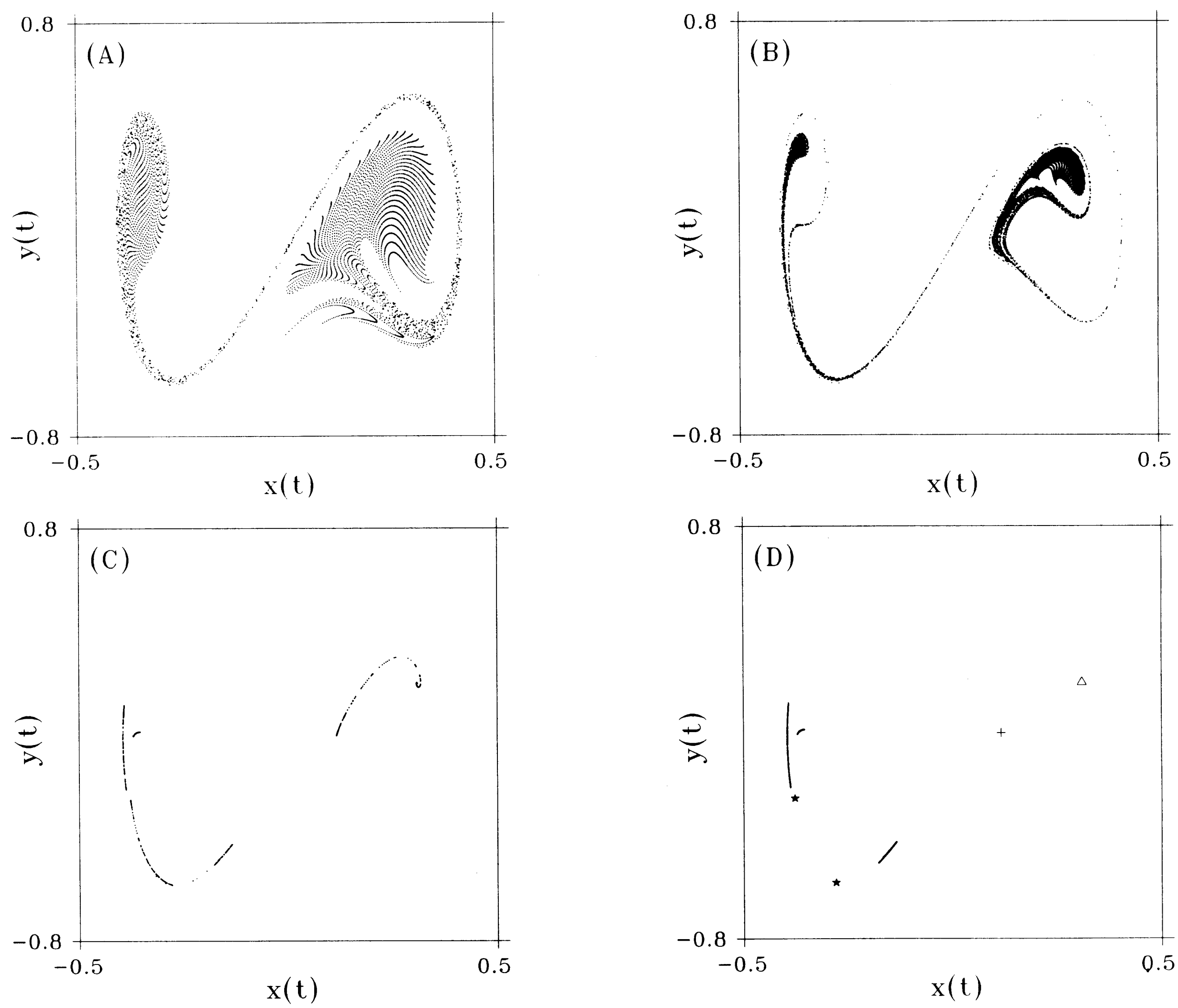

FIG. 2. The image of 80 × 80 uniformly distributed initial points after time (a) $t = T$, (b) $t = 2T$, (c) $t = 20T$, (d) $t = 70T$ (T is the period of the sinusoidal drive). The last picture coincides with the plot of four coexisting attractors: The triangle and the cross correspond to the two different attractors of period T, respectively. Two stars represent the period $2T$ attractor and heavy dots refer to the three piece chaotic attractor. All four plots are obtained for the unperturbed system (Δ and ϵ are equal to zero).

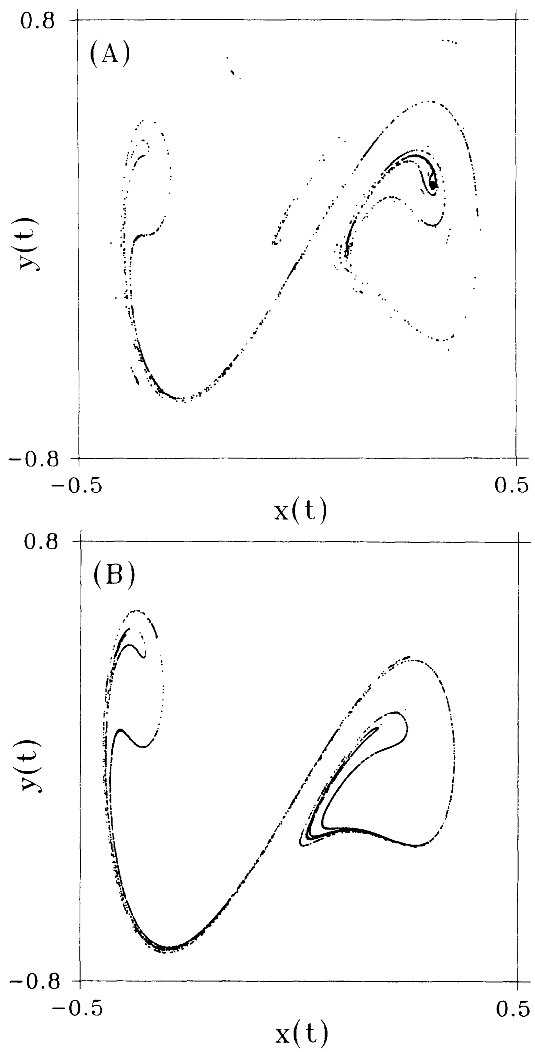

FIG. 3. (a) A piece of one long trajectory generated by the periodically kicked Duffing oscillator; see text for details; (b) the chaotic attractor generated by the unperturbed Duffing oscillator but for slightly different parameters: $\lambda = 0.795$, $T = 1.7943$, $\gamma = 1.025$, $a = 10$, $b_0 = 0.25$, and $c = 120$.

Fig. 2(d). The situation changes drastically when we start to kick the system periodically. The parameters of the perturbing sequence are equal: the amplitude $\Delta = 17$, the period of perturbation $\tau = 10T$, and the duration of a single kick $\tau_k = \epsilon T$ with $\epsilon = 0.08$. In Fig. 3(a) a part of the long trajectory generated by the periodically kicked Duffing oscillator is shown. The total evolution time $t_{tot} = 45\,000T$ but only the middle part of the trajectory for time t satisfying $20\,000T < t < 25\,000T$ is plotted in Fig. 3(a). For other time intervals the corresponding plots look very similar. Thus, the numerical experiments seem to suggest that a long time behavior of the perturbed system is connected to the nonattracting set which exists in the unperturbed system; see Fig. 2 and Fig. 3(a). Moreover, this set appears to be very close to the chaotic attractor which exists in the free unperturbed system for other control parameters; see Fig. 3(b).

In order to check that the perturbed oscillator may generate a chaotic trajectory, we integrate a few pairs of trajectories. Each pair starts from two different initial points and the distance bewteen these points is very small. After a suitable long time ($t_{tot} \approx 150T$) the endpoints of each two trajectories are completely uncorrelated.

The permanent behavior connected with the nonattracting chaotic set presented here is only one of many possible reactions of the dynamical system on the periodic perturbation. For example, for other parameters of the perturbing sequence we can observe long transient chaos which looks very similar to the permanent evolution shown in Fig. 3(a). This transient ends on the final periodic attractor which may correspond to one of the original (unperturbed) attractors or it may correspond to a new attractor which exists only in the perturbed system.

In summary we want to stress that the role of nonattracting sets is certainly not less important than the role of attractors. Both kinds of sets may be responsible for the long time behavior of a system. A typical repellor is usually characterized by an infinite hierarchy of unstable periodic orbits. The nonattracting set investigated here consists of only a few unstable periodic orbits and their unstable manifolds. In spite of this, this rather uncomplicated set may be connected with quite complicated chaotic evolution of a periodically kicked oscillator.

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